Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Numerical Analysis

Chapter 3
Finite Difference Method

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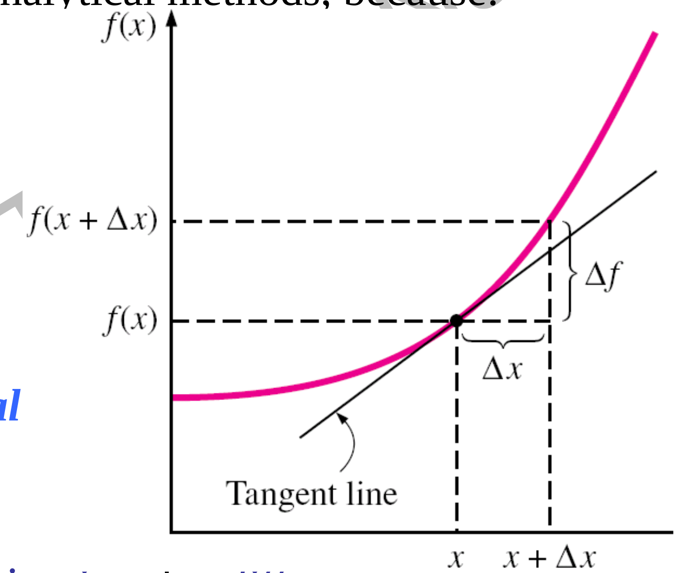
What is a Finite Difference Equation?

A finite difference method (FDM) is is one of the available numerical methods for solving differential equations by approximating derivatives with finite differences. Both the spatial domain and time interval (if applicable) are discretized, or broken into a finite number of steps.

In general real life EM problems cannot be solved by using the analytical methods, because:

1. The PDE is not linear,
2. The solution region is complex,
3. The boundary conditions are of mixed types,
4. The boundary conditions are time dependent,



The numerical methods are based on replacing the ***differential equations*** by ***algebraic equations (***finite difference equations).

For **finite difference** method, this is done by replacing the *derivatives* by *differences*.

A function *f* that depends on *x*.

The **first derivative** of *f*(*x*) at a point is equivalent to the *slope* of a line tangent to the curve at that point.

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)) - f(x)}{\Delta x}$$

If we don't take the indicated limit, we will have the following *approximate* relation for the derivative:

$$\frac{df(x)}{dx} \cong \frac{f(x+\Delta x)) - f(x)}{\Delta x}$$

The differential $\frac{d(f)}{dx} \approx \frac{\Delta f}{h}$ the difference

Finite Difference Approximation Of The Derivative

Steps of finite difference solution:

- Divide the solution region into a grid of nodes,
- Approximate the given differential equation by finite difference equivalent,
- Solve the differential equations subject to the boundary conditions and/or initial conditions.

The rate of the change of the function with respect to the variable *x* is accounted between the

current value at: $x = x_i$

and the step forward at $x_{i+1} = x_i + \Delta x$

f_{i+1} at $x = x_{i+1}$	f_i at $x = x_i$	f_{i-1} at $x = x_{i-1}$

$x_{i+1} = x_i + \Delta x$, and $x_{i-1} = x - \Delta x$,

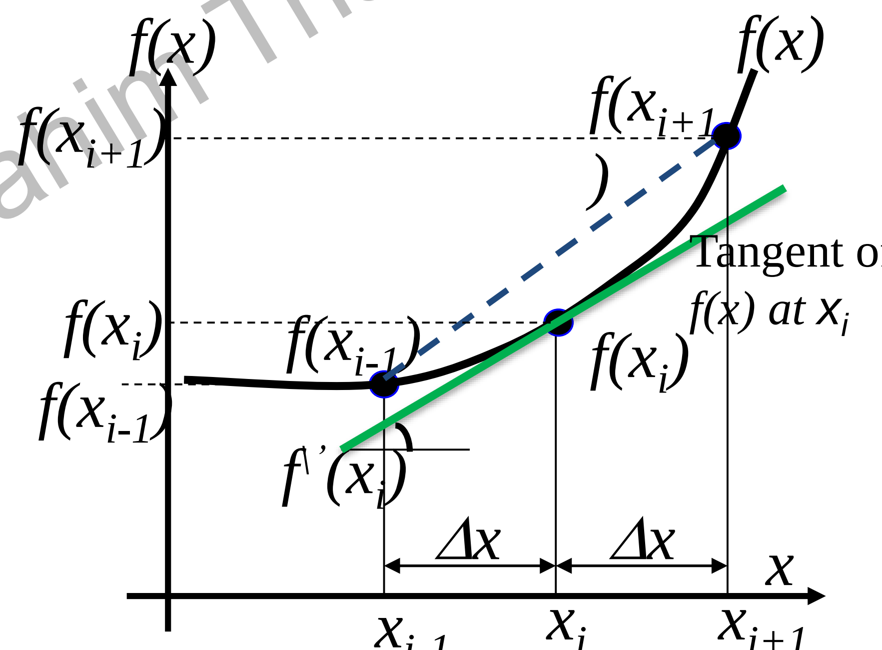
The derivative $f'(x)$ of a function $f(x)$

at the point $x = x_i$ is defined by:

$$\left|\frac{d(f)}{dx}\right|_{x=x_i} = \lim_{\Delta x \to 0} \frac{f(x_i+\Delta x)) - f(x_i)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_i+\Delta x)) - f(x_i)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{\Delta f}{h}$$

where $h = \Delta x$



The differential $\frac{d(f)}{dx} \approx \frac{\Delta f}{h}$ the difference

It is important to be aware of the fact that smaller the steps Δx the closer the values between the differential and difference of the rate change of the function

Three Basic Finite Difference Methods

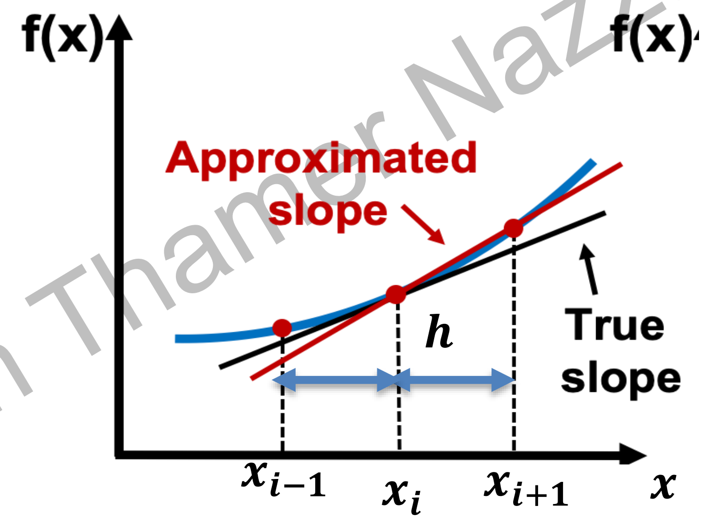
In **finite difference** approximations of the slope, we can use values of the function in the neighborhood of the point to achieve the goal. There are 3 ways to express differentials of a function:

1. Forward difference is the rate of change of the function values between the "current" step at x_i, and the function value at a step "forward" at x_{i+1} and equal to the slope of the line that connects points: $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$:

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i} = \frac{f(x_{i+1})-f(x_i)}{h}$$

$$\left.\frac{d(f)}{dx}\right|_{x=x_i+1} = \frac{f(x_{i+2})-f(x_{i+1})}{h}$$

$$\left.\frac{d(f)}{dx}\right|_{x=x_i+2} = \frac{f(x_{i+3})-f(x_{i+2})}{h}$$

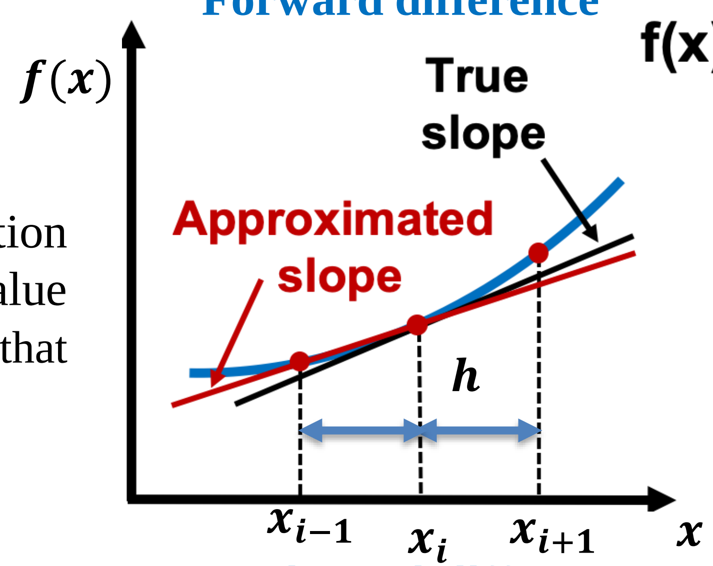


Forward difference

2. Backward difference is the rate of change of the function values between the "current" step at x_i, and the function value at a step "back" at x_{i-1}, which is the slope of the line that connects points: $((x_i, f(x_i))$ and $(x_{i-1}, f(x_{i-1}))$:

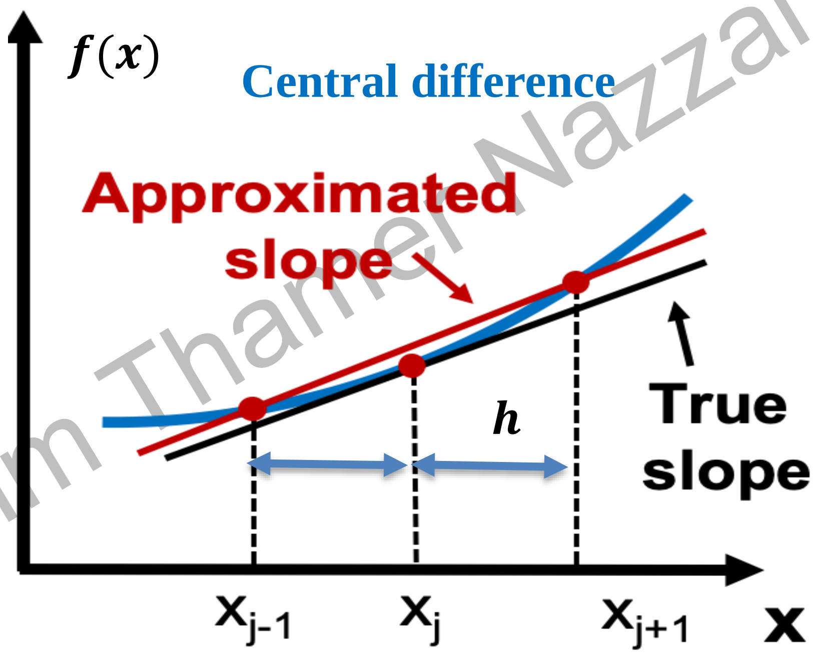
$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}$$

$$= \frac{f(x_i)-f(x_{i-1})}{h}$$



Backward difference

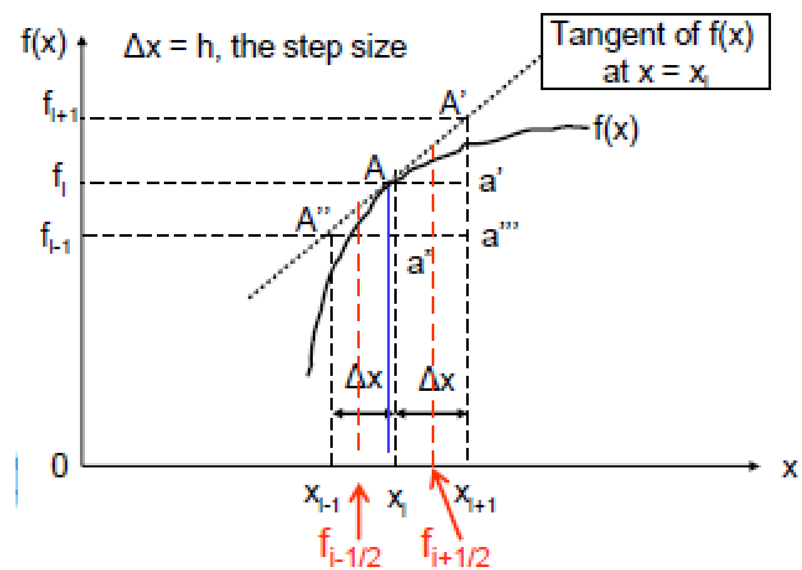
3. Central difference is the rate of change of function *f(x)* is accounted for between the step at back at *(x-Δx)* and the step ahead of x, i.e. *(x+Δx), which* is the slope of the line that connects points:$(x_{i-1}, f(x_{i-1}))$ and $(x_{i+1}, f(x_{i+1}))$:



$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

The step size in above is "*2h*" which is "big" in compromising the accuracy.

A more accurate "central difference scheme" is to reduce the step size in each forward and backward direction by half as show



We will have the corresponding expressions:

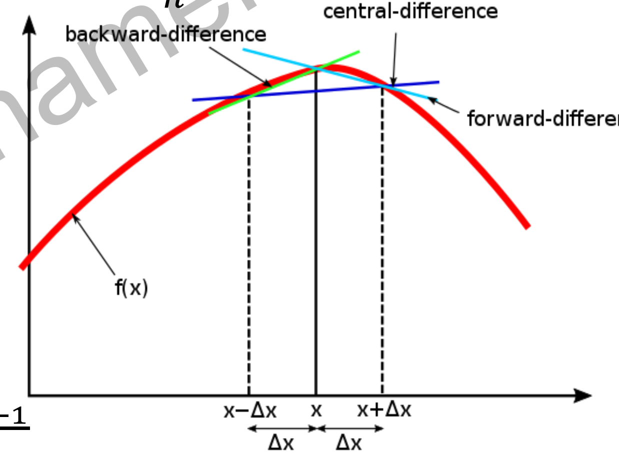
$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f\left(x_{i+\frac{1}{2}}\right) - f\left(x_{i-\frac{1}{2}}\right)}{h}$$

1. The forward difference scheme

The second order derivative of the function at x_i can be derived by the following procedure

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_i} = \frac{d}{dx}\left(\frac{d(f)}{dx}\right)\Bigg|_{x=x_i} = \lim_{\Delta x\to 0}\frac{\nabla f|_{x_{i+1}} - \nabla f|_{x_i}}{\Delta x} \approx \frac{\nabla f|_{x_{i+1}} - \nabla f|_{x_i}}{\Delta x} = \frac{\nabla f|_{x_{i+1}} - \nabla f|_{x_i}}{h}$$

$$\frac{\frac{f(x_{i+2})-f(x_{i+1})}{h} - \frac{f(x_{i+1})-f(x_i)}{h}}{h} = \frac{f(x_{i+2})-2f(x_{i+1})+f(x_i)}{h^2}$$



2. The backward difference scheme

and the send order derivatives in the form:

$$\frac{d}{dx}\left(\frac{d(f)}{dx}\right)\Bigg|_{x=x_i} = \lim_{\Delta x\to 0}\frac{\nabla f|_{x_i} - \nabla f\big|_{x_{i-1}}}{\Delta x} \approx \frac{\nabla f|_{x_i} - \nabla f\big|_{x_{i-1}}}{\Delta x} = \frac{\nabla f|_{x_i} - \nabla f\big|_{x_{i-1}}}{h}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_i} = \frac{d}{dx}\left(\frac{d(f)}{dx}\right)\Bigg|_{x=x_i} = \frac{\frac{f(x_i)-f(x_{i-1})}{h} - \frac{f(x_{i-1})-f(x_{i-2})}{h}}{h} = \frac{f(x_i)-2f(x_{i-1})+f(x_{i-2})}{h^2}$$

3. The central difference scheme

$$\frac{d}{dx}\left(\frac{d(f)}{dx}\right)\Bigg|_{x=x_i} = \lim_{\Delta x\to 0}\frac{\nabla f|_{x_{i+\frac{1}{2}}} - \nabla f\big|_{x_{i-\frac{1}{2}}}}{\Delta x} \approx \frac{\nabla f|_{x_{i+\frac{1}{2}}} - \nabla f\big|_{x_{i-\frac{1}{2}}}}{\Delta x} = \frac{\nabla f\big|_{x_{i+\frac{1}{2}}} - \nabla f\big|_{x_{i-\frac{1}{2}}}}{h}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_i} = \frac{d}{dx}\left(\frac{d(f)}{dx}\right)\Bigg|_{x=x_i} = \frac{\frac{f(x_{i+1})-f(x_i)}{h} - \frac{f(x_i)-f(x_{i-1})}{h}}{h} = \frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}$$

Example : Comparing numerical and analytical differentiation.

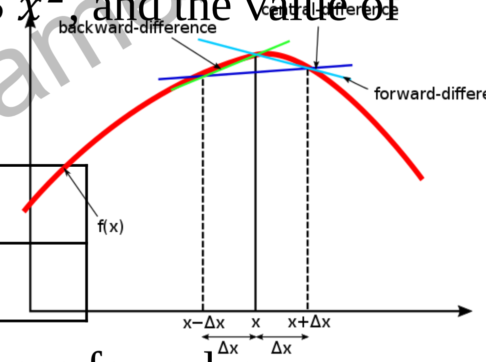
Consider the function $f(x) = x^3$.Calculate its first derivative at point $x = 3$ numerically with the forward, backward, and central finite difference formulas and using:

(a) Points $x = 2$, $x = 3$, and $x = 4$. (b) Points $x = 2.75$, $x = 3$, and $x = 3.25$.

Compare the results with the exact (analytical) derivative.

SOLUTION

Analytical differentiation: The derivative of the function is $f'(x) = 3\,x^2$, and the value of the derivative at $x = 3$ is $f'(3) = 3\,(3^2) = 27$.



The points used for numerical differentiation are:

x	2	3	4
$f(x)$	8	27	64

Using the derivatives the forward, backward, and central finite difference formulas are:

Forward finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_i)}{h} \Rightarrow \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(4)-f(3)}{4-3} = \frac{64-27}{1} = 37 \quad error = \left|\frac{37-27}{27} \times 100\right| = 37.04\%$$

Backward finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_i)-f(x_{i-1})}{h} \Rightarrow \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(3)-f(2)}{3-2} = \frac{27-8}{1} = 19 \quad error = \left|\frac{19-27}{27} \times 100\right| = 29.63\%$$

Central finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \Rightarrow \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(4)-f(2)}{4-2} = \frac{64-8}{2} = 28 \quad error = \left|\frac{28-27}{27} \times 100\right| = 3.704\%$$

(b)The points used for numerical differentiation are:

x	2.75	3	3.25
$f(x)$	2.75^3	27	3.25^3

Using the derivatives the forward, backward, and central finite difference formulas are:

Forward finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_i)}{h} \quad \Rightarrow \quad \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(3.25)-f(3)}{3.25-3} = \frac{3.25^3-27}{0.25} = 29.3125$$

$$error = \left|\frac{29.3125 - 27}{27} \times 100\right| = 8.565\%$$

Backward finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_i)-f(x_{i-1})}{h} \quad \Rightarrow \quad \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(3)-f(2.75)}{3-2.75} = \frac{27-2.75^3}{0.25} = 24.8125$$

$$error = \left|\frac{24.8125 - 27}{27} \times 100\right| = 8.102\ \%$$

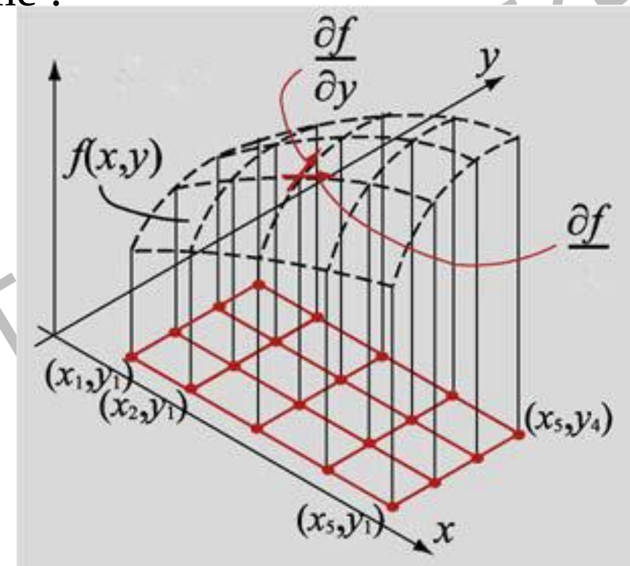
Central finite difference

$$\left.\frac{d(f)}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \quad \Rightarrow \quad \left.\frac{d(f)}{dx}\right|_{x=3} = \frac{f(3.25)-f(2.75)}{3.25-2.75} = \frac{3.25^3-2.75^3}{0.5} = 27.0625$$

$$error = \left|\frac{27.0625 - 27}{27} \times 100\right| = 0.2315\%$$

Finite Difference Schemes of partial derivatives

For a function *f(x,y)* with two independent variables, the partial derivatives with respect to *x* and *y* at the point (x_i, y_i) can be approximated into three basic scheme :

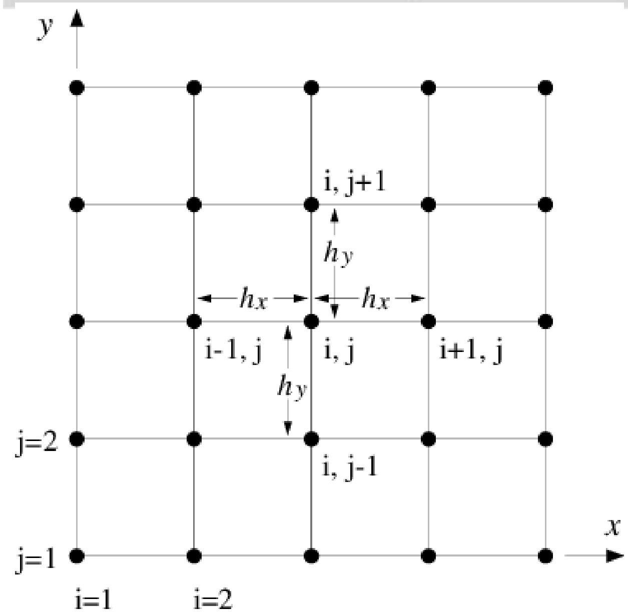


1. Forward difference is the rate of change of the function values between the “current” step at (x_i, y_j), and the function value at a step “forward” at(x_{i+1}, y_j)):

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i\\y=y_j}} = \frac{f(x_{i+1},y_j)-f(x_i,y_j)}{x_{i+1}-x_i} = \frac{f(x_{i+1},y_j)-f(x_i,y_j)}{h_x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_i\\y=y_j}} = \frac{f(x_i,y_{j+1})-f(x_i,y_j)}{y_{j+1}-y_j} = \frac{f(x_i,y_{j+1})-f(x_i,y_j)}{h_y}$$

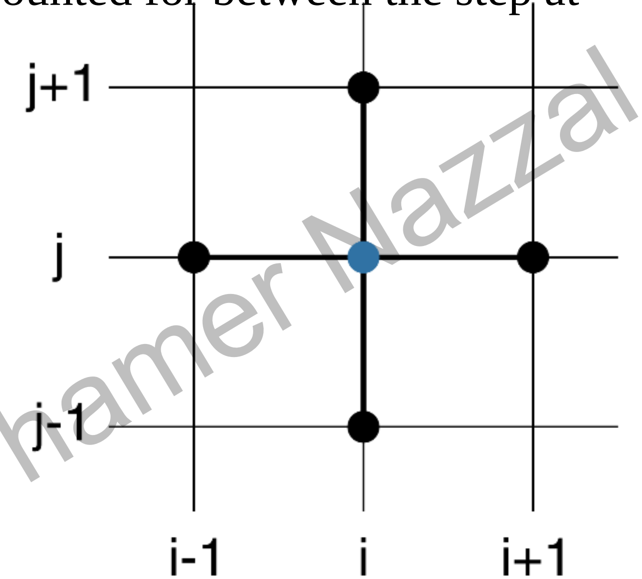
2. Backward difference is the rate of change of the function values between the “current” step at x_i, y_{j_i}, and the function value at a step “back” at x_{i-1}, y_j, :



$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i\\y=j}} = \frac{f(x_i,y_j)-f(x_{i-1},y_j)}{x_i-x_j-1} = \frac{f(x_i,y_j)-f(x_{i-1},y_j)}{h_x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_i\\y=y_j}} = \frac{f(x_i,y_j)-f(x_i,y_{j-1})}{y-y_{j-1}} = \frac{f(x_i,y_j)-f(x_i,y_{j-1})}{h_y}$$

3. Central difference is the rate of change of function *f(x)* is accounted for between the step at back at (x_{i-1}, y_j) and the step ahead of (x_{i+1}, y_j) $for\ \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i \\ y=y_j}}$

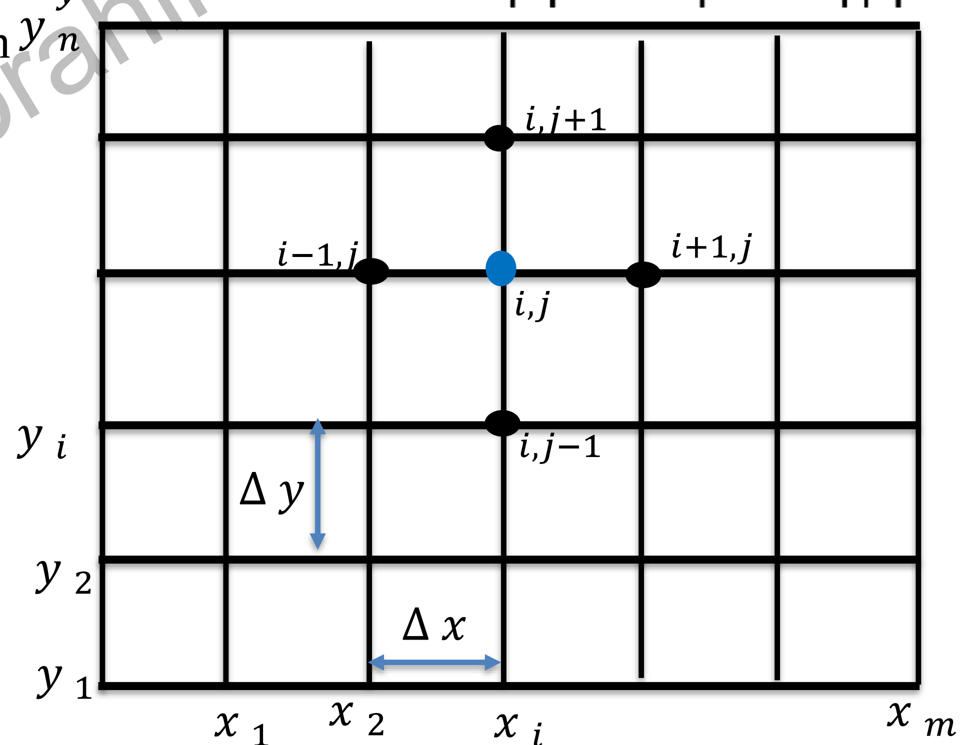


$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i \\ y=y_j}} = \frac{f(x_{i+1,}y_j)-f(x_{i-1,}y_j)}{x_{i+1}-x_{i-1}} = \frac{f\left(x_{i+1,}y_j\right)-f\left(x_{i-1,}y_j\right)}{2h_x}$$

And is accounted for between the step atback at (x_i, y_{j-1}) and the step ahead of (x_i, y_{j+1}) $for\ \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_i \\ y=y_j}}$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_i \\ y=y_j}} = \frac{f(x_i,y_{j+1})-f(x_i,y_{j-1})}{y_{j+1}-y_{j-1}} = \frac{f\left(x_i, y_{j+1}\right)-f\left(x_i, y_{j-1}\right)}{2h_y}$$

The step size in above is "*2h*" which is "big" in compromising the accuracy.



A more accurate "central difference scheme" is to reduce the step size in each forward and backward direction by half as show

We will have the corresponding expressions:

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_i \\ y=y_j}} = \frac{f\left(x_{i+\frac{1}{2}}, y_j\right)-f\left(x_{i-\frac{1}{2}}, y_j\right)}{h_x}$$

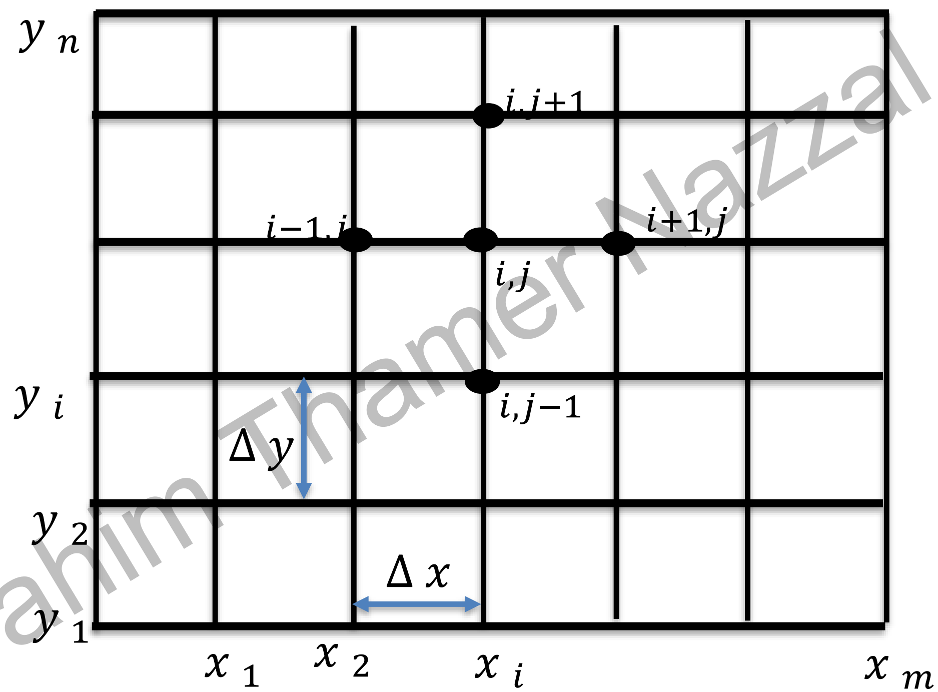
$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_i \\ y=y_j}} = \frac{f\left(x_i, y_{j+\frac{1}{2}}\right)-f\left(x_i, y_{j-\frac{1}{2}}\right)}{h_y}$$

The second order derivative

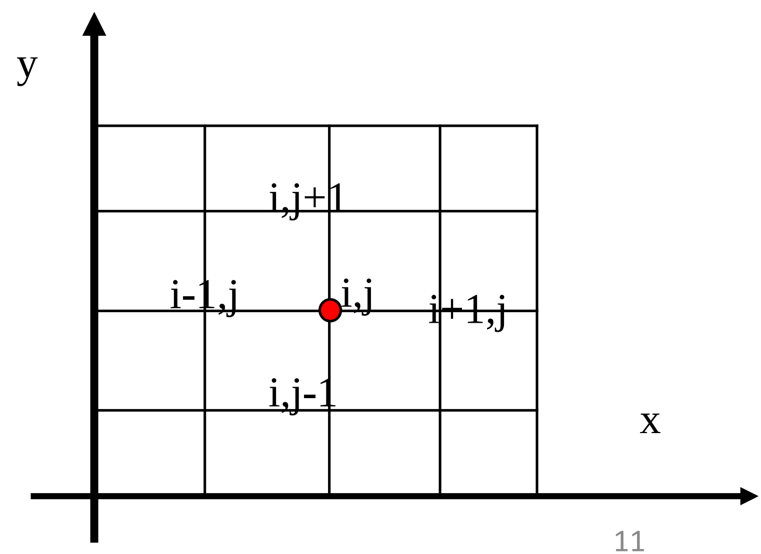
The second order derivative of the function at *x and y* can be derived by the following

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=x_i\\ y=y_j}} = \frac{f(x_{i+1},y_j)-2f(x_i,y_j)+f(x_{i-1},y_j)}{h^2{}_x}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\substack{x=x_i\\ y=y_j}} = \frac{f(x_i,y_{j+1})-2f(x_i,y_j)+f(x_i,y_{j-1})}{h^2{}_y}$$



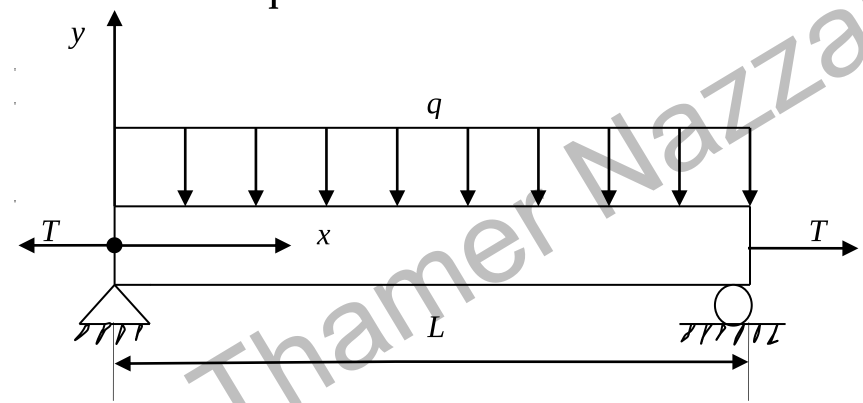
$$\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{\substack{x=x_i\\ y=y_j}} = \frac{[f(x_{i+1},y_{j+1})-f(x_{i-1},y_{j+1})]-[f(x_{i+1},y_{j-1})-f(x_{i-1},y_{j-1})]}{2h^2{}_x h^2{}_y}$$



Example 1

The deflection y in a simply supported beam with a uniform load q and a tensile axial load T is given by

$$\frac{d^2y}{dx^2} - \frac{Ty}{EI} = \frac{qx(L-x)}{2EI}$$



Where
X = location along the beam (in).
T = tension applied (lbs)
E = Young's modulus of elasticity of the beam (psi).
I= second moment of area (in^4)
Q= uniform loading intensity (lb/in)
L= length of beam (in)

B C

$$y(x=0)=0 \qquad y(x=L)=0$$

Figure 3 Simply supported beam for Example

Given, $T = 7200$ lb, $q = 5400$ lbsin, $L = 75\,\text{in}$ $E = 30\,\text{Msi}$, and $I = 120\,\text{in}^4$

a) Find the deflection of the beam at $x = 50''$ Use a step size of $\Delta x = 25''$ and approximate the derivatives by central divided difference approximation.

Solution

a) Substituting the given values,

$$\frac{d^2y}{dx^2} - \frac{7200y}{(30\times10^6)(120)} = \frac{(5400)x(75-x)}{2(30\times10^6)(120)}$$

$$\frac{d^2y}{dx^2} - 2\times10^{-6}\,y = 7.5\times10^{-7}\,x(75-x) \qquad (1)$$

Approximating the derivative $\frac{d^2y}{dx^2}$ at node i by the central divided difference approximation,

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

$$\frac{d^2y}{dx^2} - 2\times10^{-6}y = 7.5\times10^{-7}x(75-x)$$

We can rewrite the equation as

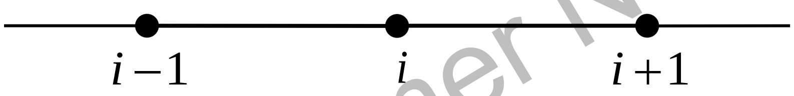


Figure central difference method.

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} - 2\times10^{-6}y_i = 7.5\times10^{-7}x_i(75-x_i)$$

$$\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

Since $\Delta x = 25$, we have 4 nodes as given in Figure 3

The location of the 4 nodes then is

$$x_1 = 0$$

$$x_2 = x_1 + \Delta x = 0 + 25 = 25$$

$$x_3 = x_2 + \Delta x = 25 + 25 = 50$$

$$x_4 = x_3 + \Delta x = 50 + 25 = 75$$

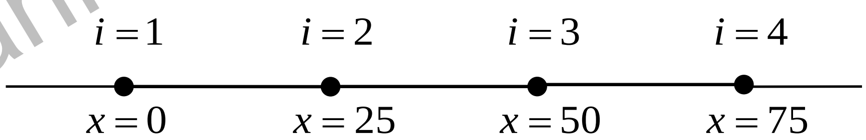


Figure Finite difference method from x= 0 to x=75 with $\Delta x = 25$.

Writing the equation at each node, we get

Node 1: From the simply supported boundary condition at $x= 0$ we obtain

$$y_1 = 0$$

Node 2: Rewriting equation (1) for node 2 gives

$$\frac{y_3 - 2y_2 + y_1}{(25)^2} - 2\times10^{-6}y_2 = 7.5\times10^{-7}x_2(75-x_2)$$

$$0.0016y_1 - 0.003202y_2 + 0.0016y_3 = 7.5\times10^{-7}(25)(75-25)$$

$$0.0016y_1 - 0.003202y_2 + 0.0016y_3 = 9.375\times10^{-4}$$

Node 3: Rewriting equation (E1.4) for node 3 gives

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} - 2\times10^{-6} y_i = 7.5\times10^{-7} x_i (75 - x_i)$$

$$\frac{y_4 - 2y_3 + y_2}{(25)^2} - 2\times10^{-6} y_3 = 7.5\times10^{-7} x_3 (75 - x_3)$$

$$0.0016y_2 - 0.003202y_3 + 0.0016y_4 = 7.5\times10^{-7}(50)(75-50)$$

$$0.0016y_2 - 0.003202y_3 + 0.0016y_4 = 9.375\times10^{-4}$$

Node 4: From the simply supported boundary condition at $x = 75$, we obtain $y_4 = 0$

$$y_1 = 0$$
$$0.0016y_1 - 0.003202y_2 + 0.0016y_3 = 9.375\times10^{-4}$$
$$0.0016y_2 - 0.003202y_3 + 0.0016y_4 = 9.375\times10^{-4}$$
$$y_4 = 0$$

Above Equations are 4 simultaneous equations with 4 unknowns and can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0016 & -0.003202 & 0.0016 & 0 \\ 0 & 0.0016 & -0.003202 & 0.0016 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.375\times10^{-4} \\ 9.375\times10^{-4} \\ 0 \end{bmatrix}$$

The above equations can be solved using on of the iterative methods such as the Gauss-Siedel method). Solving the equations we get,

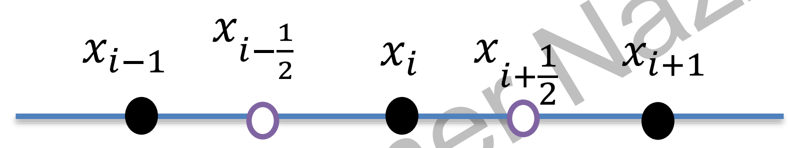
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5852 \\ -0.5852 \\ 0 \end{bmatrix}$$

$$y(50) = y(x_2) \approx y_2 = -0.5852''$$

Example :- Solve the following differential equation where $x\dfrac{d^2y}{dx^2} + y = 0$

Subject to boundary condition $y(1) = 1, y(2) = 2$

Solution-



$$\frac{\partial^2 f}{\partial x^2} = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

we know that $0 \le i \le n$, $\quad x_i = 1 + ih$, $\quad nh = 1$, $\quad$ *and let* $n = 4$,

by substituting into the main equation, one obtain

$$x_i \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = 0 \qquad n = 4 = \frac{1}{h},$$

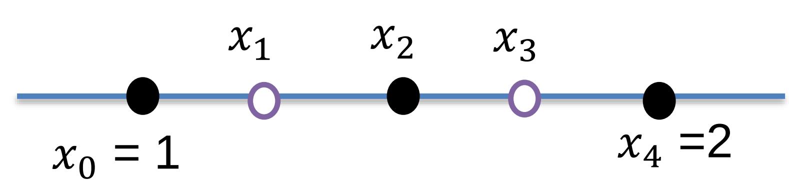
$$16\, x_i\, y_{i+1} + (1 - 32x_i)y_i + 16\, x_i y_{i-1} = 0 \qquad (1)$$

Substituting (i= 1,2,3) into the above equation, one obtain

$39\, y_1 - 20\, y_2 + 0\, y_3 = 20$

$24\, y_1 - 47\, y_2 + 24\, y_3 = 0$

$0\, y_1 - 28\, y_2 + 55 y_3 = 56$



Using gauss elimination method , one obtain

$y_1 = 1.3512$, $y_2 = 1.63495$ $\quad y_3 = 1.85053$

Discretization methods (Finite Difference) using Taylor Series Expansion

- Taylor's series expansion:

Consider a continuous function of x, namely, $f(x)$, with all derivatives defined at $x+\Delta x$

Then, the value of f at a location can be estimated from a Taylor series expanded about point x, that is

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(\Delta x)^3 + \ldots + \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(\Delta x)^n + \ldots$$

In general, to obtain more accuracy, additional higher-order terms must be included.

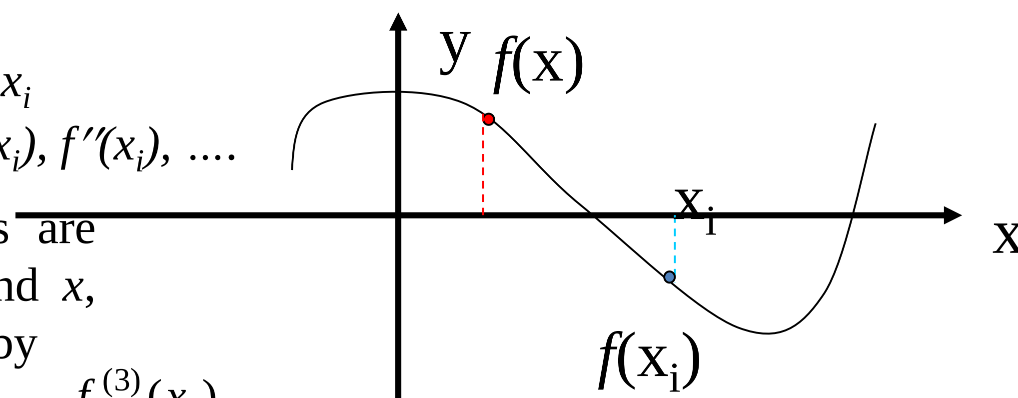
- Higher order derivatives are unknown and can be dropped when the distance between grid points is small.

Taylor Series

Problem: For a smooth function $f(x)$,

Given: Values of $f(x_i)$ and its derivatives at x_i

Find out: Value of $f(x)$ in terms of $f(x_i)$, $f'(x_i)$, $f''(x_i)$, ….



If the function f and its $n+1$ derivatives are continuous on an interval containing x_i and x, then the value of the function f at x is given by

$$f(x) = f(x_i) + f'(x_i)(x-x_i) + \frac{f''(x_i)}{2!}(x-x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x-x_i)^3$$

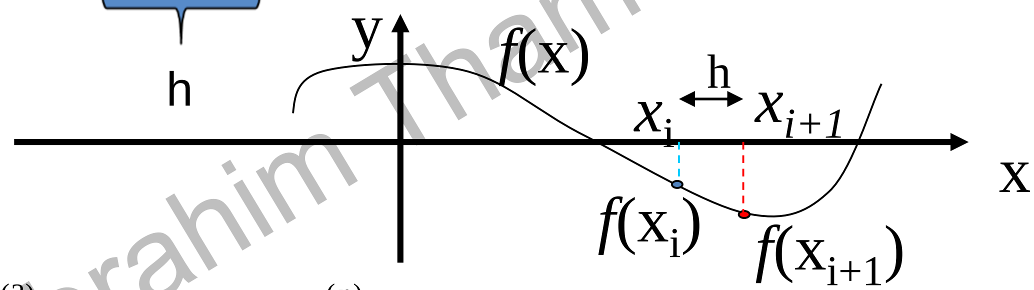
$$+ \ldots + \frac{f^{(n)}(x_i)}{n!}(x-x_i)^n + R_n$$

Finite Difference Approximations of the First Derivative using the Taylor Series (forward difference)

Assume we can expand a unction *f(x)* into a Taylor Series about the point x_{i+1}

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1}-x_i) + \frac{f''(x_i)}{2!}\underbrace{(x_{i+1}-x_i)}_{h}^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1}-x_i)^3$$

$$+ \ldots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1}-x_i)^n + R_n$$



$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + \cdots$$

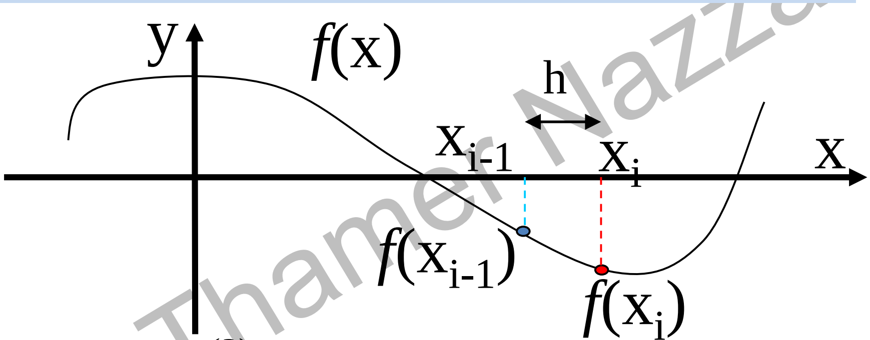
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + \frac{f''(x_i)}{2!}h + \frac{f^{(3)}(x_i)}{3!}h^2 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^{n-1} + \cdots$$

Ignore all of these terms

$$\boxed{f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}} \quad + \ 0(h)$$

Finite Difference Approximations of the First Derivative using the Taylor Series (backward difference)

Assume we can expand a function *f(x)* into a Taylor Series about the point x_{i-1}



$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}\underbrace{(x_{i-1} - x_i)^2}_{-h} + \frac{f^{(3)}(x_i)}{3!}(x_{i-1} - x_i)^3$$

$$+ \ldots + \frac{f^{(n)}(x_i)}{n!}(x_{i-1} - x_i)^n + R_n$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + \frac{f''(x_i)}{2!}h - \frac{f^{(3)}(x_i)}{3!}h^2 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^{n-1} + \cdots$$

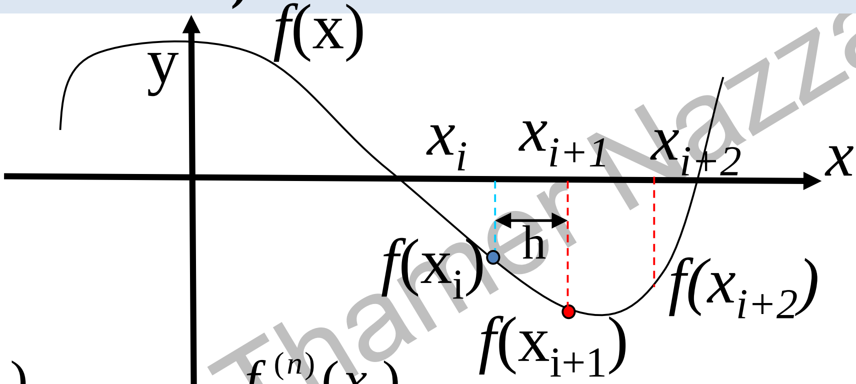
$$\boxed{f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h} + O(h)}$$

First backward difference $\longrightarrow$

$$\nabla f_i = f(x_i) - f(x_{i-1})$$

$$f'(x_i) \approx \frac{\nabla f_i}{h} + O(h)$$

Finite Difference Approximations of the Second Derivative using the Taylor Series (forward difference)



$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + \cdots \quad (1)$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)2h + \frac{f''(x_i)}{2!}4h^2 + \frac{f^{(3)}(x_i)}{3!}8h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}2^n h^n + \cdots \quad (2)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} - h\, f^{(3)}(x_i)$$

(eq. 2)- 2* (eq.1)

$$f''(x_i) = \frac{\Delta^2 f_i}{h^2} + O(h) = \frac{\Delta(\Delta f_i)}{h^2} + O(h)$$

Recursive formula for any order derivative $\longrightarrow$ $$\left.\frac{d^2 f}{dx^n}\right|_{x=x_i} = \frac{\Delta^n f_i}{h^2} + O(h)$$

Second Derivative Centered Difference Approximation (central difference)

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \quad (1)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \quad (2)$$

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)h^2 + 2\frac{f^{(4)}(x_i)}{4!}h^4 + \cdots \qquad (1)+(2)$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \cancel{+ 2\frac{f^{(4)}(x_i)}{4!}h^2 + \cdots}$$

$$\boxed{f''(x_i) \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)}$$

Centered Difference Approximation

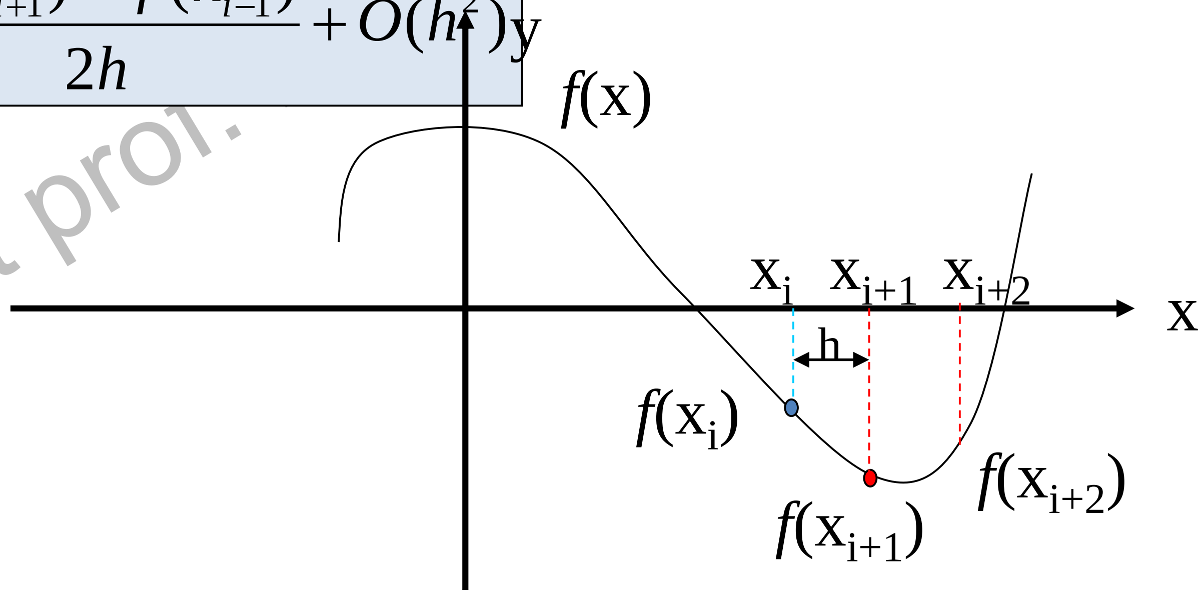
$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \qquad (1)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \qquad (2)$$

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)h + 2\frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \qquad (1)\text{-}(2)$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + \cancel{2\frac{f^{(3)}(x_i)}{3!}h^2 + \cdots}$$

$$\boxed{f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)}$$



Higher Order Finite Difference Approximations

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + \frac{f''(x_i)}{2!}h + \frac{f^{(3)}(x_i)}{3!}h^2 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^{n-1} + \cdots$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} - hf^{(3)}(x_i) + \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + \frac{\left[\dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h} + hf^{(3)}(x_i) + \ldots\right]}{2!}h$$

$$+ \frac{f^{(3)}(x_i)}{3!}h^2 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^{n-1} + \cdots$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} - \frac{h^2}{3}f'''(x) + \ldots$$

$$\boxed{f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)}$$

First Derivative		
Method	Formula	Truncation Error
Two-point forward difference	$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$	*0(h)*
Three-point forward difference	$f'(x_i) = \frac{-3\, f(x_i)+4\, f(x_{i+1})-f(x_{i+2})}{2h}$	$0(h^2)$
Two-point central difference	$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$	0(h)
Two-point backward difference	$f'(x_i) = \frac{f(x_i)-f(x_{i-1})}{h}$	$0(h^2)$
three-point central difference	$f'(x_i) = \dfrac{f(x_{i-2}) - 4\, f(x_{i-1}) + 3\, f(x_i)}{2h}$	$0(h^2)$
Four-point central difference	$f'(x_i) = \frac{f(x_{i-2})-8\, f(x_{i-1})+8\, f(x_{i+1})-f(x_{i+2})}{12h}$	$0(h^4)$

Second Derivative		
Method	Formula	Truncation Error
Three-point forward difference	$f''(x_i) = \frac{f(x_i) - 2\,f(x_{i+1}) + f(x_{i+2})}{h^2}$	0(h)
Four-point forward difference	$f''(x_i) = \frac{2\,f(x_i) - 5\,f(x_{i+1}) + 4f(x_{i+2}) - f(x_{i+3})}{h^2}$	$0(h^2)$
Three-point backward difference	$f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2}$	0(h)
Four -point backward difference	$f''(x_i) = \frac{-3\,f(x_{i-3}) + 4\,f(x_{i-2}) - 5\,f(x_{i-1}) + 2\,f(x_i)}{h^2}$	$0(h^2)$
Three-point central difference	$f''(x_i) = \frac{f(x_{i-1}) - 2\,f(x_i) + f(x_{i+1})}{h^2}$	$0(h^2)$
Five-point central difference	$f''(x_i) = \frac{-\,f(x_{i-2}) + 16\,f(x_{i-1}) - 30\,f(x_{i+1}) - 16f(x_{i+1}) - f(x_{i+2})}{12h^2}$	$0(h^4)$

Finite Difference Methods for Solving PDE's

A partial differential equation (PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives with respect to two or more independent variables. The classification of PDEs is important for the numerical solution you choose.

1 Elliptic

For example, **Laplace's equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2 Hyperbolic

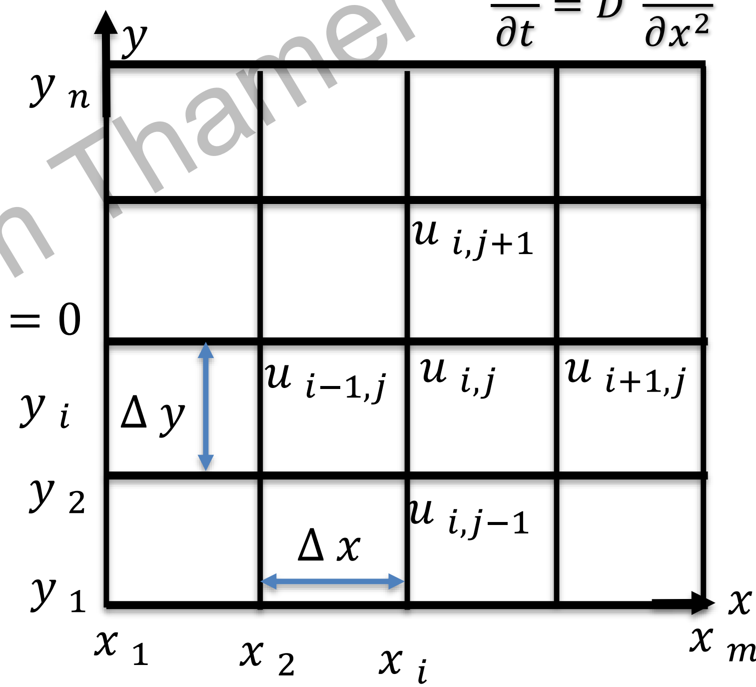
For example the 1-D wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

3 Parabolic

For example, the heat or diffusion Equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$



Using finite difference method to solve the system

1. Discretize domain into grid of evenly spaced points.
2. For nodes where u is unknown for example : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

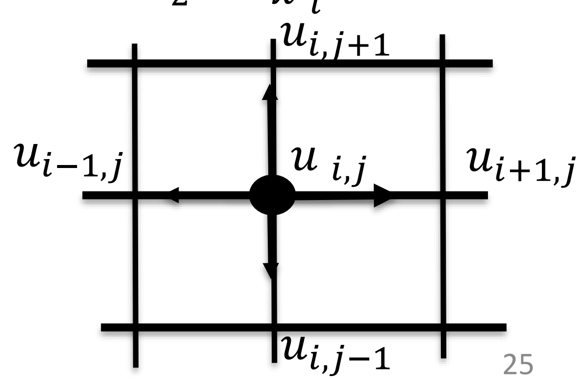
discretize

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + 0(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + 0(\Delta y^2)$$

When $\Delta x = \Delta y = h$, substitute into main eqaution

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + 0h^2 = 0$$



3. Using Boundary Conditions, write, n*m equations for $u(x_{i=1:m}, y_{j=1:n})$ or n*m unknowns.
4. Solve this banded system with an efficient scheme. Using Gauss-Seidel iteratively.

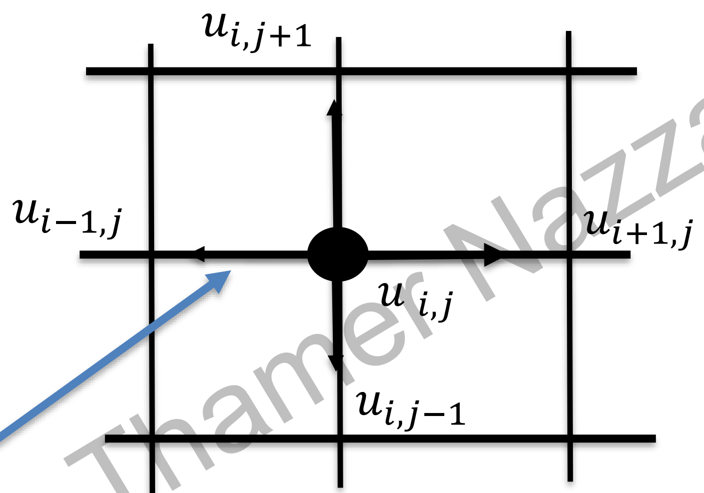
1 Elliptic equation

The Laplace Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The Laplace molecule for $\Delta x = \Delta y = h$

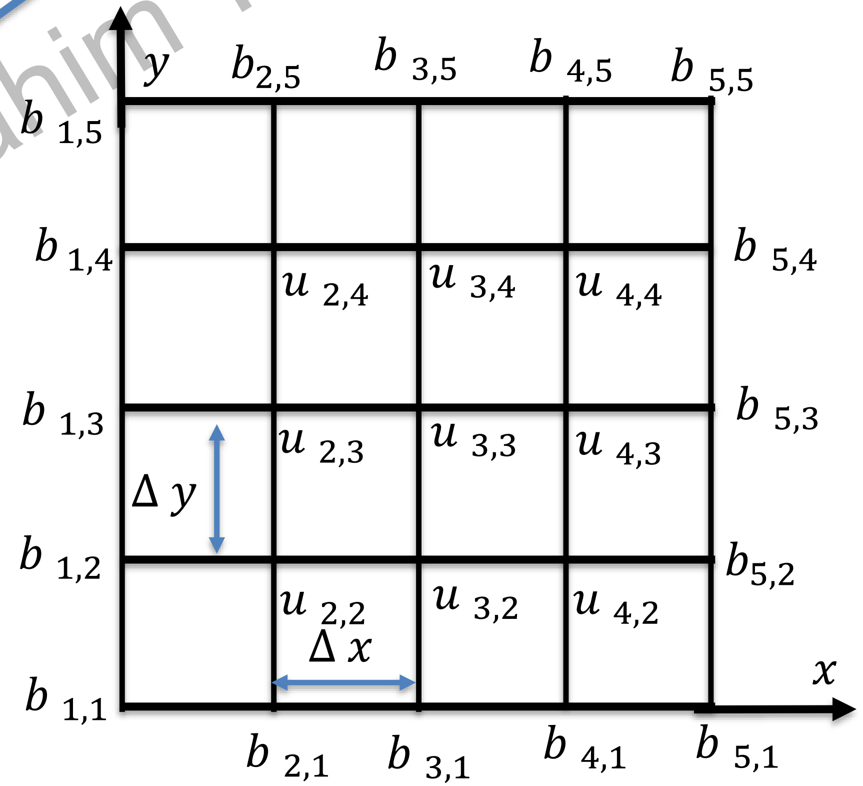
$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4\, u_{i,j} = 0$$

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$



This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. is called the **diagonal five-point formula** which is represented in Figure.

The temperature distribution can be estimated by discretizing the Laplace equation at 9 points and solving the system of linear equations.



$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2})$$

$$u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2})$$

$$u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

(*i*) **Jacobi's method.** Denoting the *n*th iterative value of $u_{i,j}$, by ${u^n}_{i,j}$, the iterative formula to solve is

$$ {u^{n+1}}_{i,j} = \frac{1}{4}({u^n}_{i+1,j} + {u^n}_{i-1,j} + {u^n}_{i,j+1} + {u^n}_{i,j-1}) $$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the *point Jacobi's formula*.

(*ii*) **Gauss-Seidal method.** In this method, the iteration formula is

$$ {u^{n+1}}_{i,j} = \frac{1}{4}({u^n}_{i+1,j} + {u^{n+1}}_{i-1,j} + {u^{n+1}}_{i,j+1} + {u^n}_{i,j-1}) $$

It utilizes the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows.

Example: A vertical steel plate of dimensions 2.7 cm X 2.7 cm and negligible thickness is in steady state conditions. On the top edge, the temperature is $100^0 C$ and on the bottom the temperature is fixed at $50^0 C$. The temperature on the left and right edges are $50^0 C$. Solve to obtain heat distribution.

The governing equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

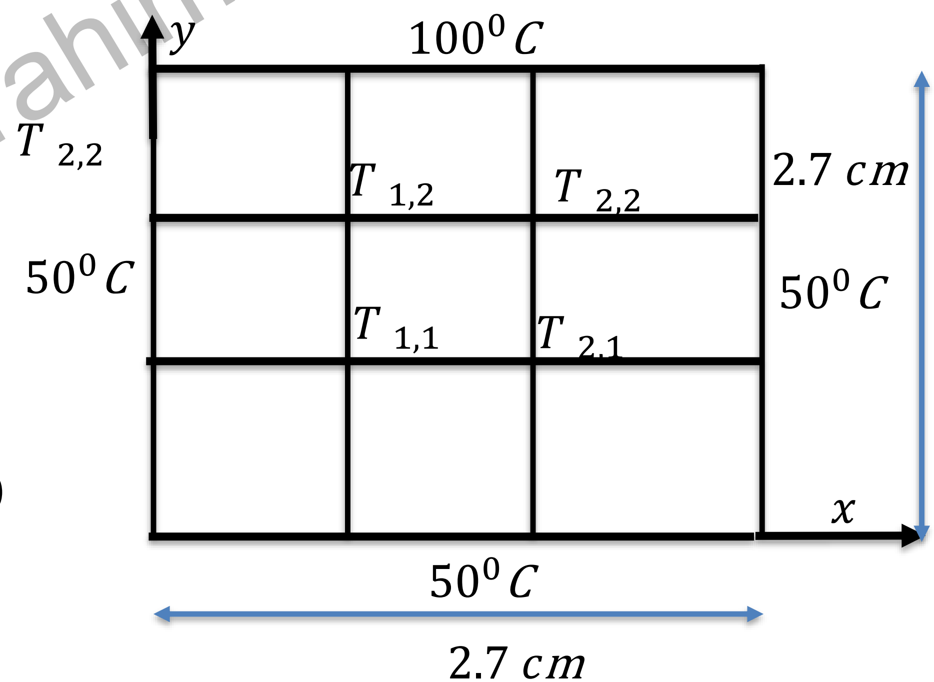
$T(x = 2.7, y) = 50^0 C;$ $T(x = 0, y) = 50^0 C;$
$T(x, y = 2.7) = 100^0 C;$ $T(x, y = 0) = 50^0 C.$

$$\Delta x = \Delta y = 0.9\ cm.$$

We know that the **diagonal five-point formula**

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

The vertical plate is discretizes below:



We need to find temperatures $T_{1,1}$, $T_{1,2}$, $T_{2,1}$ and $T_{2,2}$,,, (four unknowns)

Apply FDE at node (1,1),

$$50 + 50 + T_{1\,2} + T_{2\,1} - 4T_{1\,1} = 0$$

or

$$-4T_{1\,1} + T_{1\,2} + T_{2\,1} = -100 \qquad (1)$$

Node or grid point (1,2),

$$50 + T_{1\,1} + T_{2\,2} + 100 - 4T_{1\,2} = 0$$

or $T_{1\,1} - 4T_{1\,2} + T_{2\,2} = -150 \qquad (2)$

Node or grid point (2,1),

$$T_{1\,1} + 50 - 4\,T_{21} + T_{2\,2} + 50 = 0$$

or $T_{1\,1} - 4T_{2\,1} + T_{2\,2} = -100$ (3)

Node or grid point (2,2),

$$T_{1\,2} + T_{2\,1} - 4\,T_{22} + 100 + 50 = 0$$

$T_{1\,2} + T_{2\,1} - 4T_{2\,2} = -150$ (4)

From equations 1, 2, 3 and 4 we have

$-4T_{1\,1} + T_{1\,2} + T_{2\,1} = -100$ (1)

$T_{1\,1} - 4T_{1\,2} + T_{2\,2} = -150$ (2)

$T_{1\,1} - 4T_{2\,1} + T_{2\,2} = -100$ (3)

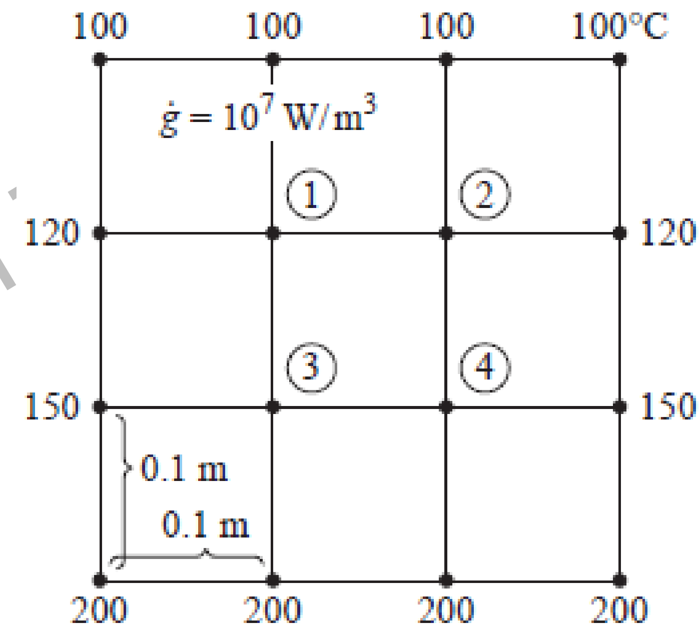
$T_{1\,2} + T_{2\,1} - 4T_{2\,2} = -150$ (4)

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_{1\,1} \\ T_{1\,2} \\ T_{2\,1} \\ T_{2\,2} \end{bmatrix} = \left[\begin{bmatrix} -100 \\ -150 \\ -100 \\ -150 \end{bmatrix}\right]$$

solve this system of linear equations.

Example Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The temperatures at the selected nodes and the thermal conditions on the boundaries are as shown. The thermal conductivity of the body is k 180 W/m · °C, and heat is generated in the body uniformly at a rate of $g = 10^7$ 10W/m3. Using the finite difference method with a mesh size of $\Delta x = \Delta y$ = 10 cm, determine the temperatures at nodes 1, 2, 3, and 4 and

Analysis: The nodal spacing is given to be $\Delta x = \Delta x = l = 0.1$ m, and the general finite difference form of an interior node equation for steady two-dimensional heat conduction for the case of constant heat generation is expressed as



$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

$$T_{left} + T_{top} + T_{bottom} + T_{right} - 4\,T_{node} + \frac{g_{node}\,l^2}{k} = 0$$

There is symmetry about a vertical line passing through the middle of the region, and thus we need to consider only half of the region. Then,

$$T_1 = T_2 \quad and \quad T_3 = T_4$$

$node$ 1 $\quad 100 + 120 + T_2 + T_3 - 4\,T_1 + \frac{gl^2}{k} = 0$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$node$ 3 $\quad 150 + 200 + T_1 + T_4 - 4\,T_3 + \frac{gl^2}{k} = 0$

noting that $T_1 = T_2$ and $T_3 = T_4$ and substituting

$$200 + T_3 - 3T_1 + \frac{0.1^2 \times 10^7}{180} = 0$$

$$350 + T_1 - 3T_3 + \frac{0.1^2 \times 10^7}{180} = 0$$

The solution of the above system is

$T_1 = T_2$ = 411.5 C and $T_3 = T_4$ = 439 C

Explicit and Implicit methods

Explicit and implicit methods are approaches used in numerical analysis for obtaining numerical approximations to the solutions of time-dependent differential equations.

Explicit Method = a formulation of equation into a FD equation that expresses **one** unknown in terms of the known values or express all future (t + Δt) values, T(x, t + Δt), in terms of current (t) and previous (t - Δt) information, which is known.

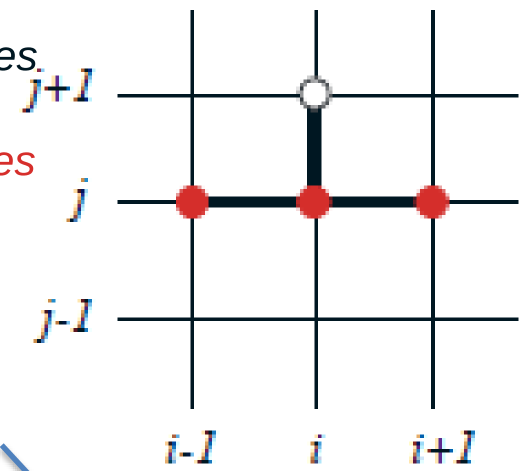
Implicit methods calculate a solution by solving an equation involving both the current state of the system and the later one or **Implicit** Schemes express all future (t + Δt) values, T(x, t + Δt), in terms of other future (t + Δt), current (t), and sometimes previous (t - Δt) information.

Finite Difference Solution of Partial Differential Equations: Parabolic Equation

- Solve the p.d.e. $$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

unknown values

known values



The finite difference approximation to the PDE is then

Evaluate the p.d.e. at *(i,j)*, *(i* = spatial index and *j* = temporal index

Depending on how $\frac{\partial u}{\partial t}$ is approximated, we have three basic schemes:

1. Explicit method **2. Implicit method** **3. Crank–Nicolson method** .

1. The explicit method - one unknown or nodal value is directly expressed in terms of known pivotal values. • ***The process advancing from a known time level(s) to the unknown time level is called "time marching".***

In this approaches using a forward difference at time t and a second order central difference for the space derivatives.

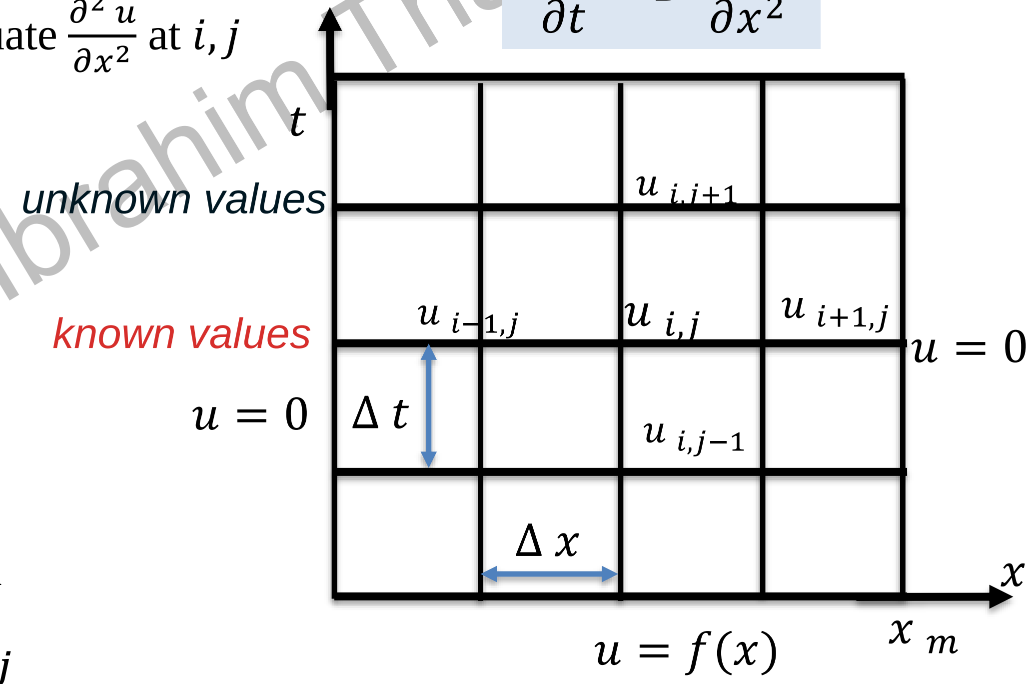
$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

Use a central difference approximation to evaluate $\frac{\partial^2 u}{\partial x^2}$ at i, j

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

- Let's use a forward difference approximation to evaluate $\frac{\partial u}{\partial t}$ at i, j

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{u(x_{i,j+1}) - u(x_{i,j})}{\Delta t}$$



Substituting into the main equation, one obtain

$$\frac{u_{i,j}+1-u_{i,j}}{\Delta t} = \mathrm{D}\,\frac{u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

$$u_{i,j+1} = u_{i,j} + \Delta t\,\mathrm{D}\,\frac{u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

Define the parameter r as $r = \frac{D\,\Delta t}{(\Delta x)^2}$

$$(u_{i,j+1} - u_{i,j}) = r\,(u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j})$$

$$u_{i,j+1} = r\,u_{i+1,j} + [1-2r]u_{i,j} + r\,u_{i-1,j}$$

We can write out the matrix system of equations we will solve numerically for the temperature u.

Finite difference method PDE example (heat equation)

$u_{i,j+1} = r\,u_{i+1,j} + [1-2r]u_{i,j} + r\,u_{i-1,j}$ can be written as $u_{j+1} = \mathrm{A}\,u_j$

$$A = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1-2r \end{bmatrix}, \qquad U \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_J \end{bmatrix}.$$

contact with blocks of melting ice and that the initial temperature distribution in non-dimensional form (unitless) is

i) $u = 0$ at $x = 0$ and $x = 1$, $t > 0$ **(the boundary condition)** $0 < x < 1$

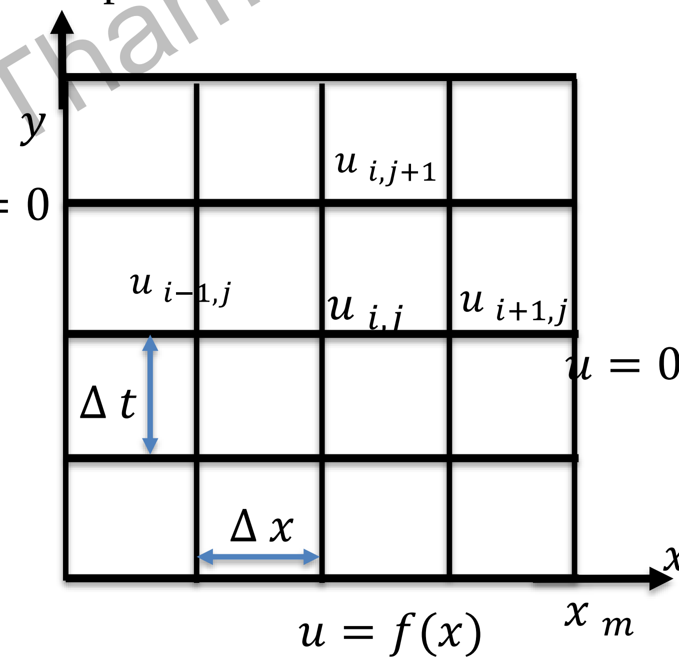
ii) $u = 2x$, for $0 \leq x \leq \frac{1}{2}$ $t = 0$ **(the initial condition)**

$u = 2(1 - x)$, for $\frac{1}{2} \leq x \leq 1$

Solve the heat conduction equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ using the Explicit Method

$\Delta x = h = 0.1 \quad \Delta t = 0.001$

$r = \frac{\Delta t}{(\Delta x)^2} = \frac{1}{10}$



solution

$u_{0,0} = 0$, $u_{1,0} = 0.2$, $u_{3,0} = 0.6$, $u_{2,0} = 0.4$ $u_{4,0} = 0.8$, $u_{5,0} = 1$,
$u_{6,0} = 0.8$, $u_{7,0} = 0.6$ $u_{8,0} = 0.4$, $u_{9,0} = 0.2$, $u_{10,0} = 0$

$$u_{i,j+1} = r\, u_{i+1,j} + [1 - 2r]u_{i,j} + r\, u_{i-1,j} \qquad r = \frac{1}{10}$$

$$u_{i,j+1} = \frac{1}{10} u_{i+1,j} + 0.8\, u_{i,j} + \frac{1}{10} u_{i-1,j}$$

when $j = 0$

$$u_{1,1} = \frac{1}{10} u_{2,0} + 0.8\, u_{1,0} + \frac{1}{10} u_{0,0} = 0.2$$

$$u_{2,1} = \frac{1}{10} u_{3,0} + 0.8\, u_{2,0} + \frac{1}{10} u_{1,0} = 0.4$$

$$u_{3,1} = \frac{1}{10} u_{4,0} + 0.8\, u_{3,0} + \frac{1}{10} u_{2,0} = 0.6$$

$$u_{4,1} = \frac{1}{10} u_{5,0} + 0.8\, u_{4,0} + \frac{1}{10} u_{3,0} = 0.8$$

$$u_{5,1} = \frac{1}{10} u_{6,0} + 0.8\, u_{5,0} + \frac{1}{10} u_{4,0} = 1$$

$$u_{6,1} = \frac{1}{10} u_{7,0} + 0.8\, u_{6,0} + \frac{1}{10} u_{5,0} = 0.8$$

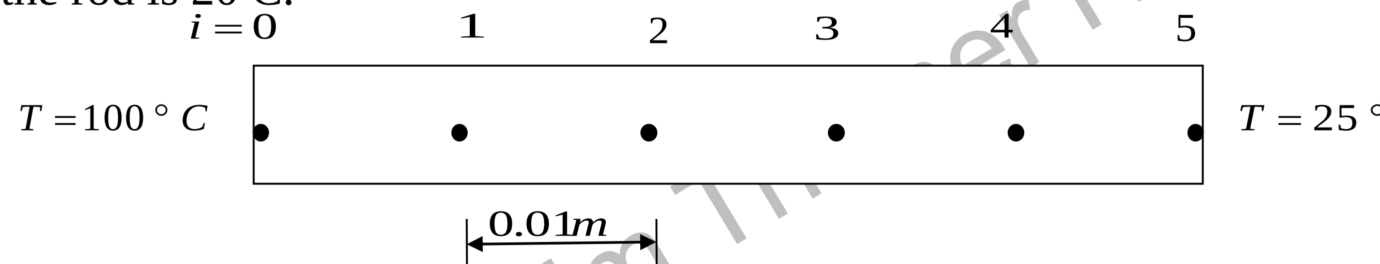
	i=0 x=0	i=1 0,1	i=2 0,2	i=3 0,3	i=4 0,4	i=5 0,5	i=6 0,6
(i=0)t=0,000	0	0,2000	0,4000	0,6000	0,8000	1,000	0,8000
(i=1)= 0,001	0	0,2000	0,4000	0,6000	0,8000	0,9600	0,8000
(i=2)= 0,002	0	0,2000	0,4000	0,6000	0,7960	0,9280	0,7960
(i=3)= 0,003	0	0,2000	0,4000	0,5996	0,7896	0,9016	0,7896
(i=4)= 0,004	0	0,2000	0,4000	0,5986	0,7818	0,8792	0,7818
(i=5)= 0,005	0	0,2000	0,3999	0,5971	0,7732	0,8597	0,7732
(i=10)= 0,01	0	0,1996	0,3968	0,5822	0,7281	0,7867	0,7281
. . .							
(i=20)= 0,02	0	0,1938	0,3781	0,5373	0,6486	0,6891	0,6486

Example: Consider a steel rod that is subjected to a temperature of 100 C on the left end and 25 C on the right end. If the rod is of length 0.05 m, use the explicit method to find the temperature distribution in the rod from $t = 0$ and $t = 9$ seconds. Use $\Delta x = 0.01$ and $\Delta t = 3\ s$.

Given: $k = 54 \frac{W}{m^2\ K}$ $\rho = 7800 \frac{kg}{m^3}$, $C = 490\ \frac{j}{kg.K}$

The initial temperature of the rod is 20 C.

Solution



Recall,

$$\alpha = \frac{k}{\rho C}$$

therefore,

$$\alpha = \frac{54}{7800 \times 490}$$

$$= 1.4129 \times 10^{-5}\ m^2 / s$$

Then,

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

$$= 1.4129 \times 10^{-5} \frac{3}{(0.01)^2} = 0.4239$$

Number of time steps,

$$= \frac{t_{final} - t_{initial}}{\Delta t}$$

$$= \frac{9-0}{3} = 3$$

Boundary Conditions .

$$\left. \begin{aligned} T_0^j &= 100°C \\ T_5^j &= 25°C \end{aligned} \right\} \text{for all } j = 0,1,2,3$$

All internal nodes are at for $t = 0 \sec$ **This can be represented as,**

$$T_i^0 = 20°C, \text{ for all } i = 1,2,3,4$$

Nodal temperatures when $t = 0\sec$ **,** $j = 0$ **:**

$$T_0^0 = 100°C$$

$$\left.\begin{aligned} T_1^0 &= 20°C \\ T_2^0 &= 20°C \\ T_3^0 &= 20°C \\ T_4^0 &= 20°C \end{aligned}\right\} \text{Interior nodes}$$

$$T_5^0 = 25°C$$

We can now calculate the temperature at each node explicitly using the equation formulated earlier, $T_i^{j+1} = T_i^j + \lambda\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right)$

Nodal temperatures when $t = 3\sec$

$\underline{i = 0}$ $T_0^1 = 100°C - \text{Boundary Condition}$

setting $j = 0$

$\underline{i = 1}$

$$\begin{aligned} T_1^1 &= T_1^0 + \lambda\left(T_2^0 - 2T_1^0 + T_0^0\right) \\ &= 20 + 0.4239\left(20 - 2(20) + 100\right) \\ &= 20 + 0.4239(80) \\ &= 20 + 33.912 \\ &= 53.912°C \end{aligned}$$

$\underline{i = 2}$

$$\begin{aligned} T_2^1 &= T_2^0 + \lambda\left(T_3^0 - 2T_2^0 + T_1^0\right) \\ &= 20 + 0.4239\left(20 - 2(20) + 20\right) \\ &= 20 + 0.4239(0) \\ &= 20 + 0 \\ &= 20°C \end{aligned}$$

Nodal temperatures when $t = 3\sec$ **,** $j = 1$ **:**

$$T_0^1 = 100°C - \text{Boundary Condition}$$

$$\left.\begin{aligned} T_1^1 &= 53.912°C \\ T_2^1 &= 20°C \\ T_3^1 &= 20°C \\ T_4^1 &= 22.120°C \end{aligned}\right\} \text{Interior nodes}$$

$$T_5^1 = 25°C - \text{Boundary Condition}$$

Nodal temperatures when $t = 6\sec$

$i = 0$ $T_0^2 = 100°C - \text{Boundary Condition}$

setting $j = 1$**,**

$i = 1$

$$T_1^2 = T_1^1 + \lambda\left(T_2^1 - 2T_1^1 + T_0^1\right)$$
$$= 53.912 + 0.4239(20 - 2(53.912) + 100)$$
$$= 53.912 + 0.4239(12.176)$$
$$= 53.912 + 5.1614$$
$$= 59.073°C$$

$i = 2$

$$T_2^2 = T_2^1 + \lambda\left(T_3^1 - 2T_2^1 + T_1^1\right)$$
$$= 20 + 0.4239(20 - 2(20) + 53.912)$$
$$= 20 + 0.4239(33.912)$$
$$= 20 + 14.375$$
$$= 34.375°C$$

Nodal temperatures when $t = 6\sec$ **,** $j = 2$ **:**

$T_0^2 = 100°C - \text{Boundary Condition}$

$$\left.\begin{aligned} T_1^2 &= 59.073°C \\ T_2^2 &= 34.375°C \\ T_3^2 &= 20.889°C \\ T_4^2 &= 22.442°C \end{aligned}\right\} \text{Interior nodes}$$

$T_5^2 = 25°C - \text{Boundary Condition}$

Nodal temperatures when $t = 9\sec$

$i = 0$ $T_0^3 = 100°C - \text{Boundary Condition}$

setting $j = 2$**,**

$i = 1$

$$T_1^3 = T_1^2 + \lambda\left(T_2^2 - 2T_1^2 + T_0^2\right)$$
$$= 59.073 + 0.4239(34.375 - 2(59.073) + 100)$$
$$= 59.073 + 0.4239(16.229)$$
$$= 59.073 + 6.8795$$
$$= 65.953°C$$

$i = 2$

$$T_2^3 = T_2^2 + \lambda\left(T_3^2 - 2T_2^2 + T_1^2\right)$$
$$= 34.375 + 0.4239(20.899 - 2(34.375) + 59.073)$$
$$= 34.375 + 0.4239(11.222)$$
$$= 34.375 + 4.7570$$
$$= 39.132°C$$

Nodal temperatures when $t = 9\,\text{sec}$, $j = 3$

$$T_0^3 = 100°C - \text{Boundary Condition}$$

$$\left.\begin{aligned} T_1^3 &= 65.953°C \\ T_2^3 &= 39.132°C \\ T_3^3 &= 27.266°C \\ T_4^3 &= 22.872°C \end{aligned}\right\} \text{Interior nodes}$$

$$T_5^3 = 25°C - \text{Boundary Condition}$$

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted below.